$\boxed{\text{Lecture 22}}$

$B = H\mathcal{U}$ $\quad H$ fixes Wx_0

$X_a = \overline{C_a}$ and C_a is a B-orbit: $C_a = Bax_0 = \mathcal{U}ax_0$
(but $By \neq \mathcal{U}y$ for most pts)

$$\mathfrak{b} = \mathfrak{h} \oplus \overbrace{\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha}^{\mathfrak{n}=\mathrm{Lie}(\mathcal{U})}$$

$X^a = \overline{C^a}$ where $C^a = B^-$-orbit of the same pt, $X^a = B^- a x_0$

$\mathfrak{b}^- = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha \qquad B^- = w_0 B w_0$ so $C^a = w_0 B \underline{w_0 a} x_0 = w_0 C_{w_0 a}$
and $X^a = w_0 X_{w_0 a}$.

Note that $[X^a] = [X_{w_0 a}]$ since G is connected: $[M] = [gM]$ $\forall g \in G$.

Also X_a and X^a intersect: they both contain ax_0.

<u>Thm.</u> $\langle [X_a], [X^b] \rangle = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{else.} \end{cases}$ That is, $\{[X_a]\ a \in W/W_P\}$ and $\{[X^a]\ a \in W/W_P\}$ are $\langle , \rangle$-dual bases of $H^*(G/P)$.

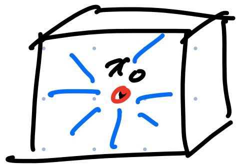
<u>Cor.</u> $\langle [X_a], [X_b] \rangle = \begin{cases} 1 & \text{if } a = w_0 b \\ 0 & \text{else} \end{cases}$

<u>Sketch of pf of thm.</u>

We'll just show X_a and X^a intersect <u>transversely</u> at ax_0. $\boxed{\text{For } G/B}$

$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ How do $\mathcal{U}, \mathcal{U}^-$ act near x_0? $\mathcal{U}$ trivially (rk 0)
$+$
$\mathcal{U}^-$ freely (rk N)
N.

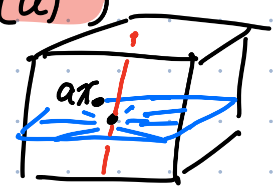
$T_{x_0} G/B \cong \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}^-$



What about near ax_0?

$\mathcal{U} \cong \mathcal{U}_{\Phi_a^+} \mathcal{U}_{\Gamma_a^+}$, $\mathcal{U}_{\Phi_a^+}$ freely $\mathcal{U}_{\Gamma_a^+}$ trivially (rk $\ell(a)$)

Γ: stay $+$ when moved through a
Φ: become $-$ when moved through a.



$\mathcal{U}^- \cong \mathcal{U}_{-\Phi_a^+} \mathcal{U}_{-\Gamma_a^+}$, $\mathcal{U}_{-\Phi_a^+}$ trivially (rk $N - \ell(a) = \ell(w_0 a)$)

Summary: One can write $\mathcal{U}^-$ as $\mathcal{U}_1^- \mathcal{U}_2^-$ s.t. C_a, C^a are $a\cdot(\mathcal{U}_1^- x_0)$ and $a.(\mathcal{U}_2^- x_0)$. $\mathbb{C}^N$ $\mathbb{C}^\ell \oplus \mathbb{C}^{N-\ell}$

Thus the chart $\mathcal{U}^- \to G/B$ is a transverse slice chart for C_a, C^a.
$u \longmapsto auB$ $\square$

Exercise. In $\mathbb{CP}^2$, W/W_P has a single element of length 1, call it m. Then $\ell(m)=1$. $m = w_0 m$.

X_m = a projective line in $\mathbb{CP}^2$ (so $X_m \cap X_{w_0 m} = X_m \cap X_m = X_m$!)

X^m = a different projective line in $\mathbb{CP}^2$.

Exercise. In $\underbrace{\mathrm{Flag}}_{\text{view as } \{(p,\ell)\ p \text{ point of } \mathbb{CP}^2 \subset \ell \text{ line}\}}(\mathbb{C}^3)$, the following is an example of $X_a \cap X^a$ = point:

$$X_a = \{(p,\ell) \mid p = [e_1]\}$$
$$X^a = \{(p,\ell) \mid [e_3] \in \ell\}$$

$X_a \cap X^a = ([e_1], [e_1 e_3])$

whereas

$$X_a = \{(p,\ell) \mid p = [e_1]\}$$
$$X_{w_0 a} = \{(p,\ell) \mid [e_1] \in \ell\}$$

and $X_a \subset X_{w_0 a}$!

Exercise. Flag$(\mathbb{C}^4)$ has a middle dimension. Describe a basis of the middle-dim H^i of the form $a_1 \ldots a_n, b_1, \ldots, b_n$ where $\langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0$ and $\langle a_i, b_j\rangle = \delta_{ij}$.

Borel's description of cohomology of G/B, G/P.

$K \subset G$ compact real form, built from root data of (G,H).

$T := K \cap B = K \cap H \cong (S^1)^r \subset (\mathbb{C}^*)^r \cong H$ $\qquad \mathfrak{h} \overset{\mathbb{R}\text{-vec sp}}{=} \mathfrak{a} \oplus \mathfrak{t}$

$G/B \cong K/T$

Let V be a vec space $/k$. $S(V) := \bigoplus_{i \geq 0} \Sigma^i V$ is a k-algebra $(+, \otimes)$

$\Sigma^i V$ — symmetric tensor power

$\mathcal{P} := S(\mathfrak{t}^*)$ polynomial ring assoc to (G, B, H), $\mathfrak{t}^*$ vec space $/\mathbb{R}$.

If $t_1, \ldots, t_r$ is a basis of $\mathfrak{t}$, then $x_1, \ldots, x_r$ = coord fns rel to $t_1, \ldots t_r$ is a basis of $\mathfrak{t}^*$ and $\mathcal{P} \cong k[x_1, \ldots, x_r]$.

e.g. $\mathfrak{g} = \mathfrak{sl}_n \mathbb{C}$. $\mathfrak{k} = \mathfrak{su}(n)$ $\mathfrak{t}$ = pure imag diag matrices trace zero.

Let $x_k \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_n \end{pmatrix} = a_k$. Then $\mathcal{P} \cong k[x_1, \ldots, x_n] / (x_1 + \cdots + x_n)$

$$\cong k[x_1, \ldots, x_{n-1}]$$

W acts on $\mathfrak{h}$ preserving $\mathfrak{a}, \mathfrak{t}$

$\rightsquigarrow$ W acts on $\mathfrak{t}^*$ (contragredient / adjoint) linearly

$\Rightarrow$ W acts on $\mathcal{P}$ by $\mathbb{R}$-algebra aut

$\mathrm{Sym}_n \curvearrowright k[x_1, \ldots, x_n]$ induces $\mathrm{Sym}_n \curvearrowright \mathcal{P}(SL_n\mathbb{C})$.

$\mathcal{P}^W :=$ W-invt subalgebra of $\mathcal{P}$ — the ring of invariants (symmetric polynomials)

J = ideal of $\mathcal{P}$ generated by the positive-degree elts of $\mathcal{P}^W$
(polynomials divisible by a nontrivial symmetric polynomial)

$\mathcal{R} = \mathcal{P}/J$ = the coinvariant ring (polynomials mod symmetric ones)

$\mathcal{P}, \mathcal{P}^W, J, \mathcal{R}$ are all graded where $k \in \mathbb{Z}^+$ corresp to degree k.

<u>Thm</u>. There is a natural degree-doubling ring isomorphism

$$\mathcal{R} \longrightarrow H^*(G/B).$$

<u>Cor</u>. $H^*(G/B)$ is algebraically generated by $H^2(G/B)$.